ESD-TR-69-108

Technical Note 1969-6

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8 May 1969

Prepared under Electronic Systems Division Contract AF 19(628)-5167 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

AD690994

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. The work was sponsored by the Department of the Army, Office Chief Research and Development, under Air Force Contract AF 19(628)-5167.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

OPTIMAL MISMATCHED FILTER DESIGN FOR RADAR RANGING AND RESOLUTION

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ABSTRACT

In a single-target radar environment, matched filters provide the maximum output signal-to-noise ratio for target detection and yield the minimum mean-squared error estimate of target range. In a multiple-target environment, the sidelobes of the compressed pulse must be considered in the system design because of the likelihood of false alarms. In this case, the signal processor uses weighting filters which are not matched to the transmitted waveform. In this report, expressions for the mean-squared range estimation error, the estimate bias, and the effects of the sidelobes are derived in terms of the impulse response of an arbitrary mismatched filter. We desire to find that impulse response which leads to an unbiased estimate having the minimum range estimate variance subject to preassigned resolution (i.e., sidelobe) constraints. This optimization problem is formulated in state-space in which the optimal control law is sought. Pontryagin's maximum principle is used to obtain necessary conditions for the optimum filter. When the sidelobe constraints are neglected, these conditions lead to the matched filter solution. In an attempt to synthesize the optimal filter for the general case, we set up a nonlinear programming problem involving the set of unknown Lagrange multipliers. This should be a computationally easier problem to solve than the original variational problem. An example is given which illustrates the methodology for synthesizing the optimum filter when the class of admissible controls (i.e., filters) is restricted by physical considerations. It is in this case that the real power of the state-space development is clearly demonstrated.

Accepted for the Air Force
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OPTIMAL MISMATCHED FILTER DESIGN
FOR RADAR RANGING AND RESOLUTION

I. INTRODUCTION

It has long been known that the matched filter represents the "optimum" means of processing data to obtain estimates of target range. Optimality, in this case, means that the estimates have the smallest mean-squared error possible, a result which is valid only when the signal-to-noise ratio (SNR) is large. In many applications, target resolvability is a consideration almost as important as range accuracy, in which case it is not clear that the matched filter is the best receiver to use. In fact, for some waveforms (for example, a sinusoidal pulse), very good range accuracy can be obtained but the resolution problem is significant since the envelope of the matched filter output signal has large subsidiary sidelobes.

One approach to this problem is to assume that the receiver is a matched filter and then try to design the input signal which will produce good range estimates subject to constraints on the sidelobe structure of the compressed pulse. Algorithms are now available which generate the solution to this problem,[1] but in most cases the resulting waveform is quite complicated, making it difficult to build a matched filter.

Another approach is to use both a known signal which can be transmitted easily and a mismatched filter. This was done for the linear FM waveform[2] in an effort to reduce the sidelobes of the compressed pulse, but at the expense of a loss in range accuracy. Heretofore, no effort has been made to design a mismatched filter to minimize the mean-squared range error subject to preassigned constraints on the sidelobe structure.

In this report, we assume that a given pulse is received in the presence of additive white Gaussian noise and passed through a filter which is not necessarily matched to the input pulse. We also assume that the range estimate is made by locating the time at which the envelope of the filter output achieves its peak value. The performance of this estimation scheme has been analyzed previously[3] with respect to measuring the loss of accuracy and detectability due to nonoptimum filtering. It is shown that the inability to build perfectly matched filters does result in a loss of accuracy. However, the advantages of mismatching with respect to improving the multiple-target resolution is not discussed. In general, the mismatched filter leads to biased estimates of the target range, a result which the analysis in Ref. 3 fails to take into account. Since the estimate bias can be significant, we rederive the equations describing the performance of the mismatched filter and rectify this omission.

In addition, the resolution properties of the filter are derived and we show that the composite accuracy/resolution performance of the filter depends on the filter impulse response and its first derivative. We then formulate an optimum control problem which leads to the filter impulse response resulting in the best range accuracy subject to preassigned resolution constraints.

An attractive feature of this approach to the design problem is that the class of admissible impulse responses can be restricted according to the degree of complexity one is willing to build into the receiver. For example, if tapped delay lines are to be used in the realization, then the search is performed over the tap weights and spacings. This search is then possible, taking into account the effects of tap reflections and the attenuation characteristics of the real line. This aspect of the design will be discussed in detail in a subsequent publication.

Here, we concentrate on the formulation of the optimal control problem which leads to the best mismatched filter. Using the maximum principle, we derive the matched filter when the sidelobes are ignored. Then, we restrict the impulse response to be a linear combination of known basis functions; such a realization is useful when using RC lumped parameter networks. The optimization is performed over the weight to be assigned to each function. In the Appendix, the general problem with sidelobes is analyzed and, using the maximum principle, we are able to reduce the function space optimization to a nonlinear programming problem involving the unknown multipliers.

II. SUBOPTIMAL SIGNAL PROCESSOR

We shall assume that the range of the target is to be estimated on the basis of the received pulse

$$r(t) = p(t - \tau_o) \cos(\omega_c t + \Theta) + n(t) \tag{II-1}$$

where τ_o represents the true time delay, Θ is an unknown phase introduced by channel disturbances, and n(t) is a sample function of a zero mean Gaussian random process with covariance function $E[n(t)\, n(t')] = N_o \delta(t - t')$. We assume that this signal is processed by the receiver, a block diagram of which is shown in Fig. 1. The impulse response $h(\cdot)$ is arbitrary except for the restriction that it belongs to a class of filters H.

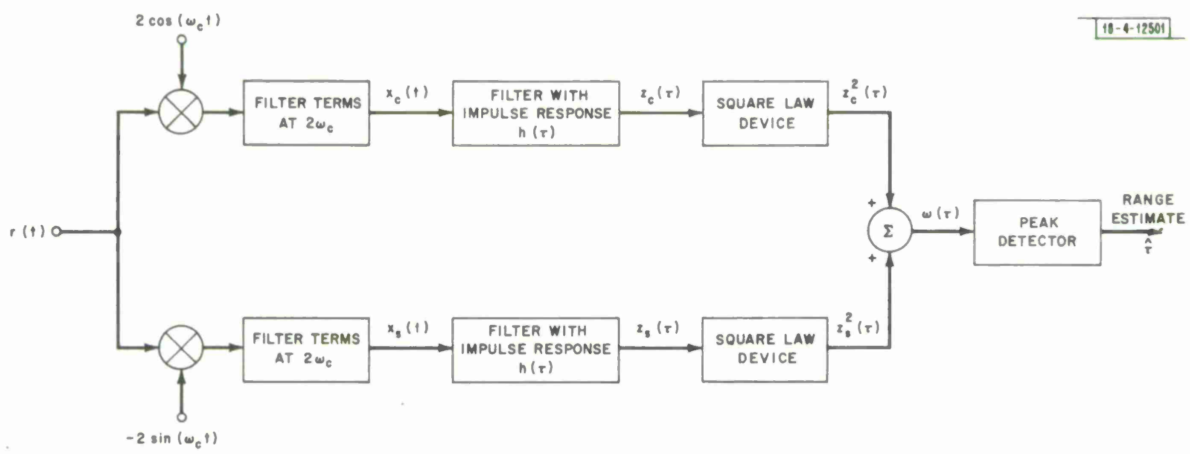

Fig. 1. Signal processor for range estimation.

Analyzing the receiver operations with respect to the waveform in Eq. (II-1), we see that

$$x_c(t) = [p(t - \tau_o) + n_c(t)] \cos\Theta - n_s(t) \sin\Theta \tag{II-2a}$$

$$x_s(t) = [p(t - \tau_o) + n_c(t)] \sin\Theta + n_s(t) \cos\Theta \tag{II-2b}$$

where $n_c(t)$ and $n_s(t)$ are the quadrature components of n(t) which can be expanded as

$$n(t) = n_c(t) \cos(\omega_c t + \Theta) - n_s(t) \sin(\omega_c t + \Theta) \tag{II-3}$$

where

$$E[n_c(t)\, n_c(t')] = E[n_s(t)\, n_s(t')] = N_o \delta(t - t') \tag{II-4a}$$

$$E[n_c(t)\, n_s(t')] = 0 \quad . \tag{II-4b}$$

The filter output signals are

$$z_c(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, x_c(t)\, dt \tag{II-5a}$$

$$z_s(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, x_s(t)\, dt \quad . \tag{II-5b}$$

Substituting Eqs. (II-2) into (II-5), we obtain

$$z_c(\tau) = [y(\tau) + \eta_c(\tau)] \cos\Theta - \eta_s(\tau) \sin\Theta \tag{II-6a}$$

$$z_s(\tau) = [y(\tau) + \eta_c(\tau)] \sin\Theta + \eta_s(\tau) \cos\Theta \tag{II-6b}$$

where

$$y(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, p(t - \tau_o)\, dt \tag{II-7a}$$

$$\eta_c(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, n_c(t)\, dt \tag{II-7b}$$

$$\eta_s(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, n_s(t)\, dt \quad . \tag{II-7c}$$

After summing the two outputs of the square-law devices, we have the signal

$$\omega(\tau) = [y(\tau) + \eta_c(\tau)]^2 + [\eta_s(\tau)]^2 \quad . \tag{II-8}$$

The receiver declares as its estimate of τ_o, the number $\hat{\tau}$ where

$$\omega(\hat{\tau}) = \max_{\tau} \omega(\tau) \quad . \tag{II-9}$$

In order for the proposed estimation scheme, Eq. (II-9), to lead to error-free performance in the absence of noise, it is necessary that

$$\max_{\tau} \omega(\tau) = \omega(\tau_o) = \max_{\tau} y^2(\tau) \tag{II-10}$$

which requires that

$$y'(\tau_o) = 0 \quad . \tag{II-11}$$

We shall see that this condition leads to unbiased estimates for the unknown time delay.

If $h(t) = p(-t)$, the receiver is the well-known matched filter processor and its performance has been well-documented in the literature.[4] The performance of the mismatched filter, $h(t) \neq p(-t)$, for estimating target range was given some attention by Hansen,[3] but an error occurred in that paper because it was assumed that the processor would generate an unbiased estimate for any filter structure. This is not true, in general; for this reason, we give our own derivation of the range accuracy formula.

In addition, we would like to point out the fact that in Ref. 3 the motivation was to determine the loss in accuracy as a result of using an imperfectly constructed matched filter, and no mention was made of the idea of synthesizing a matched filter to give good range estimates and low sidelobes simultaneously.

To proceed with the analysis of the mismatched filter (which closely follows that in Ref. 4), we assume that if the processor is to be any good then the estimate $\hat{\tau}$ should be close to τ_o, the true value, when the SNR is large. Expanding $\omega(\tau)$ about τ_o, we obtain

$$\omega(\tau) = \omega(\tau_o) + \omega'(\tau_o)(\tau - \tau_o) + \frac{1}{2}\omega''(\tau_o)(\tau - \tau_o)^2 + \dots \quad . \tag{II-12}$$

For large SNR, the higher order terms can be neglected and the maximum of $\omega(\tau)$ occurs at the point $\hat{\tau}$ where $\omega'(\tau) = 0$. Therefore,

$$\hat{\tau} - \tau_o = -\frac{\omega'(\tau_o)}{\omega''(\tau_o)} \quad . \tag{II-13}$$

From Eq. (II-8), we see that

$$\omega'(\tau) = 2[y(\tau) + \eta_c(\tau)][y'(\tau) + \eta_c'(\tau)] + 2\eta_s(\tau)\,\eta_s'(\tau) \tag{II-14}$$

$$\omega''(\tau) = 2[y'(\tau) + \eta_c(\tau)]^2 + 2[y(\tau) + \eta_c(\tau)][y''(\tau) + \eta_c''(\tau)]$$

$$+ 2[\eta_s'(\tau)]^2 + 2\eta_s(\tau)\,\eta_s''(\tau) \quad . \tag{II-15}$$

To find an expression of $(\hat{\tau} - \tau_o)$ which is first order in the noise, we express Eq. (II-14) to first order so that

$$\frac{1}{2}\omega'(\tau_o) = y(\tau_o)\,y'(\tau_o) + y(\tau_o)\,\eta_c'(\tau_o) + y'(\tau_o)\,\eta_c(\tau) \quad . \tag{II-16}$$

We shall assume that the filter has been designed for perfect noise-free performance, so that $y'(\tau_o) = 0$, which follows from our earlier discussion. Then, Eq. (II-16) becomes

$$\frac{1}{2}\omega'(\tau_o) = y(\tau_c)\,\eta_c'(\tau_o) \quad . \tag{II-17}$$

Since this is first order in the noise, it suffices to express $\omega''(\tau_o)$ to zero order in the noise. From Eq. (II-15), we find that

$$\frac{1}{2}\omega''(\tau_o) = y(\tau_o)\,y''(\tau_o) \tag{II-18}$$

where we have once again made use of the fact that $y'(\tau_o) = 0$. Then, the error in the range estimate is conveniently expressed as

$$\hat{\tau} - \tau_o = \frac{1}{-y''(\tau_o)}\,\eta_c'(\tau_o) \quad . \tag{II-19}$$

Referring to Eq. (II-7b), we see that

$$\eta'_c(\tau_o) = \int_{-\infty}^{\infty} h'(\tau_o - t)\ \eta_c(t)\ dt \tag{II-20}$$

which is a Gaussian random variable with mean zero and variance

$$\sigma^2 = N_o \int_{-\infty}^{\infty} [h'(-t)]^2\ dt \quad . \tag{II-21}$$

From Eq. (II-7a), we see that

$$y(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\ p(t - \tau_o)\ dt = \int_{-\infty}^{\infty} h(-t)\ p(t + \tau - \tau_o)\ dt \tag{II-22}$$

and, therefore,

$$y'(\tau_o) = \int_{-\infty}^{\infty} h(-t)\ p'(t)\ dt \tag{II-23a}$$

$$y''(\tau_o) = \int_{-\infty}^{\infty} h(-t)\ p''(t)\ dt \quad . \tag{II-23b}$$

Relating Eqs. (II-20) and (II-23) to (II-19), we conclude that for large SNR the proposed estimation scheme produces range estimates which are unbiased estimates of the true parameter value τ_o, and have a Gaussian distribution about τ_o. The mean-squared estimation error is

$$E(\hat{\tau} - \tau_o)^2 = N_o \cdot \frac{\int_{-\infty}^{\infty} [h'(-t)]^2\ dt}{[\int_{-\infty}^{\infty} h(-t)\ p''(t)\ dt]^2} \tag{II-24}$$

where E denotes the ensemble average. This result is valid for the class of filters for which

$$\int_{-\infty}^{\infty} h(-t)\ p'(t)\ dt = 0 \quad . \tag{II-25}$$

In fact, the proposed processor is meaningful only for this class of filters.

If the filter is matched to the signal, $h(t) = p(-t)$ and, in this case, Eq. (II-25) is always satisfied. It is then easy to show that Eq. (II-24) becomes

$$E(\hat{\tau} - \tau_o)^2 = \frac{N_o}{\int_{-\infty}^{\infty} [p'(t)]^2\ dt} \tag{II-26}$$

which is the classical result, and provides a check on our analysis to this point.

Thus far, the analysis has been restricted to the large SNR range accuracy performance of the receiver. The multiple-target resolution capabilities can be taken into account by first observing that the envelope $y^2(\tau)$ will have, in addition to a maximum at τ_o, subsidiary peaks – called sidelobes at other τ values. If these sidelobes are the same order of magnitude as $y^2(\tau_o)$, it will not be possible to distinguish between two distinct but neighboring targets. Therefore, it is desirable to design the filter to make these sidelobes small with respect to the magnitude of the central lobe at τ_o. This can be done by requiring that the constraint

$$y^2(\tau) \leqslant \epsilon(\tau)\ y^2(\tau_o)$$

be satisfied, where $\epsilon(\tau)$ represents the sidelobe constraint function and is chosen to combat the particular clutter distribution under consideration. Since

$$y(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, p(t - \tau_o)\, dt$$

$$= \int_{-\infty}^{\infty} h(-t)\, p(t + \tau - \tau_o)\, dt \qquad \text{(II-27)}$$

then $y(\tau_o)$ is independent of τ_o and the sidelobes can be kept low by requiring that

$$\left[\int_{-\infty}^{\infty} h(-t)\, p(t + \tau)\, dt\right]^2 - \epsilon(\tau) \left[\int_{-\infty}^{\infty} h(-t)\, p(t)\, dt\right]^2 \leqslant 0 \quad . \qquad \text{(II-28)}$$

In practice, $h(\cdot)$ and $p(\cdot)$ are band-limited functions, in some sense, and the continuum of constraints can be replaced by the finite number of constraints

$$\left[\int_{-\infty}^{\infty} h(-t)\, p(t + \tau_j)\, dt\right]^2 - \epsilon_j \left[\int_{-\infty}^{\infty} h(-t)\, p(t)\, dt\right]^2 \leqslant 0 \qquad j = 1, 2, \ldots, n \quad . \qquad \text{(II-29)}$$

The foregoing analysis leads to the following filter design problem: From the class admissible filters H, we wish to find that filter which minimizes the quantity

$$\frac{\int_{-\infty}^{\infty} [h'(-t)]^2\, dt}{[\int_{-\infty}^{\infty} h(-t)\, p''(t)\, dt]^2} \qquad \text{(II-30a)}$$

subject to the "zero-bias" constraint

$$\int_{-\infty}^{\infty} h(-t)\, p'(t)\, dt = 0 \qquad \text{(II-30b)}$$

and multiple-target resolution constraints

$$\left[\int_{-\infty}^{\infty} h(-t)\, p(t + \tau_j)\, dt\right]^2 - \epsilon_j \left[\int_{-\infty}^{\infty} h(-t)\, p(t)\, dt\right]^2 \leqslant 0 \qquad j = 1, 2, \ldots, n \quad . \qquad \text{(II-30c)}$$

In Sec. III, we use state-space techniques to formulate this design problem as an optimal control problem in state-space. Then, by using the maximum principle, we can derive the conditions necessary for optimality.

III. STATE-SPACE FORMULATION OF THE DESIGN PROBLEM

Here, we shall formulate the mismatched filter design problem using state-variable techniques so that the theory of optimal control can be used to synthesize the optimum filter. In order to do this, we first assume that the filter impulse response is of finite duration, T seconds, and is identically zero for $t \notin [0, T]$. In addition, we assume that the transmitted pulse p(t) is also time limited to the interval $[0, T]$, an assumption which will always be satisfied in practice. Finally, we point out the fact that in the analysis of Sec. II the performance depended on that part of the impulse response for $t < 0$. In order to guarantee the realizability of the optimum filter, we need only replace $h(-t)$ in all our equations by $h(T - t)$.

We define the control function u(t) as the first derivative of the impulse response, while the impulse response itself is set equal to the first component of the state vector; i. e.,

$$u(t) = h'(T - t) \tag{III-1}$$

$$x_1(t) = h(T - t) \tag{III-2}$$

which leads to the state equation

$$\dot{x}_1(t) = -u(t) \tag{III-3}$$

and, since the filter is initially at rest when the pulse arrives, we set

$$x_1(0) = 0 \quad . \tag{III-4}$$

In addition, we define four more state variables according to the state equations

$$\dot{x}_2(t) = h(T - t)\ p(t) \tag{III-5a}$$

$$\dot{x}_3(t) = h(T - t)\ p'(t) \tag{III-5b}$$

$$\dot{x}_4(t) = h(T - t)\ p''(t) \tag{III-5c}$$

$$\dot{x}_5(t) = [h'(T - t)]^2 \tag{III-5d}$$

each having the initial conditions

$$x_2(0) = x_3(0) = x_4(0) = x_5(0) = 0 \quad . \tag{III-6}$$

It is obvious that

$$x_2(T) = \int_0^T h(T - t)\ p(t)\ dt \tag{III-7a}$$

$$x_3(T) = \int_0^T h(T - t)\ p'(t)\ dt \tag{III-7b}$$

$$x_4(T) = \int_0^T h(T - t)\ p''(t)\ dt \tag{III-7c}$$

$$x_5(T) = \int_0^T [h'(T - t)]^2\ dt \quad . \tag{III-7d}$$

Furthermore, by substituting Eqs. (III-1) and (III-2) into (III-5) we obtain the following set of first-order differential equations

$$\dot{x}_1(t) = -u(t) \qquad x_1(0) = 0 \tag{III-8a}$$

$$\dot{x}_2(t) = p(t)\ x_1(t) \qquad x_2(0) = 0 \tag{III-8b}$$

$$\dot{x}_3(t) = p'(t)\ x_1(t) \qquad x_3(0) = 0 \tag{III-8c}$$

$$\dot{x}_4(t) = p''(t)\ x_1(t) \qquad x_4(0) = 0 \tag{III-8d}$$

$$\dot{x}_5(t) = u^2(t) \qquad x_5(0) = 0 \quad . \tag{III-8e}$$

The zero bias constraint, Eq. (II-30b), requires that

$$x_3(T) = 0 \tag{III-9}$$

which follows from the definition Eqs. (III-5b) and (III-7b). Subject to this constraint, the best range estimate is obtained by minimizing Eq. (II-30a), which is equivalent to minimizing

$$-x_4^2(T)/x_5(T) \tag{III-10}$$

which also follows from the above definitions.

Finally, the multiple-target resolution requirements are accounted for by defining the state variables

$$\dot{y}_j(t) = h(T-t)\, p(t+\tau_j) \qquad j = 1, 2, \ldots, n \tag{III-11}$$

with $y_j(0) = 0$ for all j. Then it is clear that

$$y_j(T) = \int_0^T h(T-t)\, p(t+\tau_j)\, dt \tag{III-12}$$

and for good resolution, we therefore require that

$$y_j^2(T) - \epsilon_j x_2^2(T) \leqslant 0 \qquad j = 1, 2, \ldots, n \tag{III-13}$$

which follows from Eq. (II-30c) and the above definitions. Therefore, the filter design problem is equivalent to the following optimal control problem. Find the control function u(t) and state vector $[x_1(t), \ldots, x_5(t), y_1(t), \ldots, y_n(t)]$ which

(a) satisfy the differential equations

$$\dot{x}_1(t) = -u(t)$$

$$\dot{x}_2(t) = p(t)\, x_1(t)$$

$$\dot{x}_3(t) = p'(t)\, x_1(t)$$

$$\dot{x}_4(t) = p''(t)\, x_1(t)$$

$$\dot{x}_5(t) = u^2(t)$$

$$\dot{y}_j(t) = p(t+\tau_j)\, x_1(t) \qquad j = 1, 2, \ldots, n$$

(b) satisfy the initial conditions

$$x_1(0) = \ldots = x_5(0) = 0$$

$$y_1(0) = \ldots = y_n(0) = 0$$

(c) satisfy the terminal conditions

$$x_3(T) = 0$$

$$y_j^2(T) - \epsilon_j x_2^2(T) \leqslant 0 \qquad j = 1, 2, \ldots, n$$

(d) minimize the functional

$$-x_4^2(T)/x_5(T) \quad .$$

The optimum control function is to be selected from some class of admissible controls U. This class is directly related to that of admissible impulse responses H. So far, we have implicitly assumed that if $h \in H$, it is at least once differentiable and, therefore, $u \in U$ must be piecewise continuous. Additional restrictions on U can be imposed by taking into consideration the specific application to which the filter is to be used. For example, in a radar application, it might be necessary to build the filter using a tapped delay line. The optimization would then be done on the set of tap weights, which merely requires imbedding the structure of the delay line into the above formulation. Even the physical properties of the line, such as tap reflections and line attenuation, can be handled in this manner. This represents the real power of the state-space approach, since no matter how complicated the system may become, efficient algorithms can be brought to bear on the problem. We are currently investigating this aspect of the design for linear FM pulses, and will report our findings thoroughly in a future publication.

IV. EXAMPLES OF CONDITIONS NECESSARY FOR OPTIMALITY

We shall now illustrate the control theoretical techniques which must be used to solve for the optimum control analytically. In the first example, we shall neglect the resolution constraints and allow the class U to be all piecewise continuous functions. Of course, the optimum filter will be the matched filter. The result is interesting, however, since it illustrates the fact that variational methods applied to the estimate variance for mismatched filters lead to the matched filter. Heretofore, this result has been obtained only by using statistical methods as in Ref. 4. Control techniques have been used previously to derive the matched filter,[5] but in those papers, usually only an artificially defined SNR is optimized. We have therefore obtained a rather ideal blend of the control and communication theoretical techniques.

In the second example, we restrict the class of filters H to be linear combinations of orthogonal basis functions. The optimum weighting for each component is obtained analytically, again assuming no sidelobe constraints. This result is useful in that it shows how the method applies to a smaller class of filters and, in addition, functions can be chosen which lead to a convenient RC realization of the optimum filter. Although such a filter cannot perform as well as a matched filter, it represents the best possible RC approximation to the matched filter, in the sense of minimum estimate variance. This, then, is certainly a better approach than using cut-and-try to get close to matched filter performance as was done essentially in Ref. 3.

The third example is a repeat of the first except that the sidelobe constraints are considered. Since most of the manipulations are similar to those used in the first two examples above, the details are presented in the Appendix. In this case, it is not possible to solve for the optimum control analytically. However, we show that the variational problem can be reduced to a nonlinear programming problem involving the unknown multipliers. This should be a simpler problem to solve numerically than the first. We now consider each of the examples in detail.

Example 1: No resolution constraints, arbitrary U

In this case the state equations are simply

$$\dot{x}_1(t) = -u(t) \qquad x_1(0) = 0 \tag{IV-1a}$$

$$\dot{x}_2(t) = p(t)\, x_1(t) \qquad x_2(0) = 0 \tag{IV-1b}$$

$$\dot{x}_3(t) = p'(t)\, x_1(t) \qquad x_3(0) = 0 \tag{IV-1c}$$

$$\dot{x}_4(t) = p''(t)\, x_1(t) \qquad x_4(0) = 0 \tag{IV-1d}$$

$$\dot{x}_5(t) = u^2(t) \qquad x_5(0) = 0 \quad . \tag{IV-1e}$$

Defining the functions

$$\Theta_o[\underline{x}(T)] = -x_4^2(T)/x_5(T) \tag{IV-2a}$$

$$\Theta_1[\underline{x}(T)] = x_3(T) \tag{IV-2b}$$

we want to minimize Θ_o subject to the constraint $\Theta_1 = 0$. The Hamiltonian for this problem is

$$H(\underline{x}, \underline{\lambda}, u) = -\lambda_1(t)\, u(t) + \lambda_2(t)\, p(t)\, x_1(t) + \lambda_3(t)\, p'(t)\, x_1(t)$$

$$+ \lambda_4(t)\, p''(t)\, x_1(t) + \lambda_5(t)\, u^2(t) \tag{IV-3}$$

where the costate variables $\underline{\lambda}(t)$ satisfy the equations

$$\dot{\lambda}_j(t) = -\frac{\partial H}{\partial x_j} \qquad j = 1, \ldots, 5 \quad . \tag{IV-4}$$

Therefore,

$$\dot{\lambda}_1(t) = -p(t)\, \lambda_2(t) - p'(t)\, \lambda_3(t) - p''(t)\, \lambda_4(t) \tag{IV-5a}$$

$$\dot{\lambda}_2(t) = 0 \tag{IV-5b}$$

$$\dot{\lambda}_3(t) = 0 \tag{IV-5c}$$

$$\dot{\lambda}_4(t) = 0 \tag{IV-5d}$$

$$\dot{\lambda}_5(t) = 0 \quad . \tag{IV-5e}$$

The terminal values of the costate variables are obtained from

$$\lambda_j(T) = \sum_{i=0}^{1} \alpha_i \frac{\partial \Theta_i[\underline{x}(T)]}{\partial x_j(T)} \tag{IV-6}$$

and, therefore,

$$\lambda_1(T) = 0 \tag{IV-7a}$$

$$\lambda_2(T) = 0 \tag{IV-7b}$$

$$\lambda_3(T) = \alpha_1 \tag{IV-7c}$$

$$\lambda_4(T) = -2\alpha_o x_4(T)/x_5(T) \tag{IV-7d}$$

$$\lambda_5(T) = \alpha_o x_4^2(T)/x_5^2(T) \quad . \tag{IV-7e}$$

Combining Eqs. (IV-5) and (IV-7), we get

$$\lambda_2(t) \equiv 0 \tag{IV-8a}$$

$$\lambda_3(t) \equiv \alpha_1 \triangleq \lambda_3 \tag{IV-8b}$$

$$\lambda_4(t) \equiv -2\alpha_0 x_4(T)/x_5(T) \triangleq \lambda_4 \tag{IV-8c}$$

$$\lambda_5(t) \equiv \alpha_0 x_4^2(T)/x_5^2(T) \triangleq \lambda_5 \tag{IV-8d}$$

and the equation for $\lambda_1(t)$ is simply

$$\dot{\lambda}_1(t) = -\lambda_3 p'(t) - \lambda_4 p''(t) \tag{IV-9}$$

with $\lambda_1(T) = 0$. This equation is easily integrated to give

$$\lambda_1(t) = \lambda_3[p(T) - p(t)] + \lambda_4[p'(T) - p'(t)] \quad . \tag{IV-10}$$

We shall assume that

$$p(0) = p(T) = 0 \tag{IV-11a}$$

$$p'(0) = p'(T) = 0 \tag{IV-11b}$$

so that

$$\lambda_1(t) = -\lambda_3 p(t) - \lambda_4 p'(t) \quad . \tag{IV-12}$$

The maximum principle[6] states that, for the optimum control, $\underline{\lambda}(t) \not\equiv 0$ and $\alpha_0 \leqslant 0$, and that the Hamiltonian is maximized by the optimal control. This is equivalent to maximizing

$$h(u) = -\lambda_1(t)\, u(t) + \lambda_5 u^2(t) \quad . \tag{IV-13}$$

The maximization is performed over the set of $u \in U$. At this point, the properties of the real filter are taken into account by restrictions imposed on U. For the first example, U is arbitrary and the methods of ordinary calculus can be used in the maximization. There are two cases to consider at this point: $\alpha_0 = 0$, and $\alpha_0 < 0$. If $\alpha_0 = 0$, then from Eq. (IV-8), $\lambda_4(t) \equiv 0$, $\lambda_5(t) \equiv 0$, and in Eq. (IV-12), $\lambda_1(t)$ reduces to

$$\lambda_1(t) = -\lambda_3 p(t) \quad .$$

If $\lambda_3 = 0$ or $\lambda_1(t) = 0$ for some t, then $\underline{\lambda}(t) \equiv 0$ and this contradicts the necessary condition for optimality. Therefore, $\lambda_3 \neq 0$ and maximizing Eq. (IV-13) means maximizing

$$h(u) = \lambda_3 p(t)\, u(t)$$

which is meaningless. We therefore conclude that $\alpha_0 < 0$, in which case $\lambda_5 < 0$ from Eq. (IV-8d). Therefore, Eq. (IV-13) has the well-defined maximum

$$\hat{u}(t) = \lambda_1(t)/2\lambda_5 \quad .$$

Using Eq. (IV-12), we conclude that the optimum control is of the form

$$\hat{u}(t) = -\frac{\lambda_3}{2\lambda_5}\, p(t) - \frac{\lambda_4}{2\lambda_5}\, p'(t) \quad . \tag{IV-14}$$

Thus, the optimum filter is of the form

$$\hat{x}_1(t) = \frac{\lambda_3}{2\lambda_5} \int_0^t p(\sigma)\, d\sigma + \frac{\lambda_4}{2\lambda_5} p(t) \quad . \tag{IV-15}$$

The zero bias constraint requires that $x_3(T) = 0$, which, from Eq. (IV-1c), is equivalent to

$$\int_0^T p'(t)\, \hat{x}_1(t)\, dt = 0 \quad . \tag{IV-16}$$

Applied to Eq. (IV-15), we get

$$\frac{\lambda_3}{2\lambda_5} \int_0^T p'(t) \left[\int_0^t p(\sigma)\, d\sigma\right] dt + \frac{\lambda_4}{2\lambda_5} \int_0^T p'(t)\, p(t)\, dt = 0 \quad .$$

Using integration by parts and the conditions in Eqs. (IV-11), we get

$$-\frac{\lambda_3}{2\lambda_5} \int_0^T p^2(t)\, dt = 0 \quad .$$

Since the integral represents the energy in the transmitted signal, which is nonzero, then it is necessary that $\lambda_3 = 0$. By using this result in Eqs. (IV-14) and (IV-15), the optimum control and filter are

$$\hat{u}(t) = -\frac{\lambda_4}{2\lambda_5} p'(t) \tag{IV-17a}$$

$$\hat{x}_1(t) = \frac{\lambda_4}{2\lambda_5} p(t) \quad . \tag{IV-17b}$$

The minimum cost is $-\hat{x}_4^2(T)/\hat{x}_5(T)$ where

$$\hat{x}_4(T) = \int_0^T p''(t)\, \hat{x}_1(t)\, dt$$

$$= \frac{\lambda_4}{2\lambda_5} \int_0^T p''(t)\, p(t)\, dt \tag{IV-18a}$$

$$\hat{x}_5(T) = \int_0^T \hat{u}^2(t)\, dt$$

$$= \frac{\lambda_4^2}{\lambda_5^2} \int_0^T [p'(t)]^2\, dt \quad . \tag{IV-18b}$$

Using integration by parts and Eqs. (IV-11), we can show that

$$\int_0^T p''(t)\, p(t)\, dt = -\int_0^T [p'(t)]^2\, dt \tag{IV-19}$$

and therefore the minimum cost is

$$-\int_0^T [p'(t)]^2\, dt \quad . \tag{IV-20}$$

Since this is independent of the unknown multiplier $\lambda_4/2\lambda_5$, we might as well set it equal to unity, in which case

$$\hat{u}(t) = -p'(t) \tag{IV-21a}$$

$$\hat{x}_1(t) = p(t) \quad . \tag{IV-21b}$$

However, we defined $x_1(t) = h(T-t)$. Therefore, the optimum impulse response is

$$\hat{h}(t) = p(T-t) \tag{IV-22}$$

which defines the matched filter.

Example 2: No resolution constraints, restricted U

The only difference between this and the preceding case is the class of functions U over which the Hamiltonian is to be maximized. Recall from Eq. (IV-13) that the optimum control in the set U maximized the function

$$h[u(t), t] = -\lambda_1(t)\, u(t) + \lambda_5 u^2(t) \tag{IV-23}$$

where

$$\lambda_1(t) = -\lambda_3 p(t) - \lambda_4 p'(t) \quad . \tag{IV-24}$$

In this case, we set

$$U = \left\{ u : u(t) = \sum_{j=1}^{m} c_j \psi_j(t) \right\} \tag{IV-25}$$

where $\{\psi_j\}_{j=1}^{m}$ are orthonormal, i.e.,

$$\int_0^T \psi_j(t)\, \psi_k(t)\, dt = \delta_{jk} \qquad \text{for all } j, k \quad . \tag{IV-26}$$

We shall use vector notation with $(\cdot)^t$ denoting the vector transpose. Then, if $\hat{u} \in U$ maximizes Eq. (IV-23) for each t (i.e., $\hat{u}$ is the optimal control), then

$$h[u(t), t] \leqslant h[\hat{u}(t), t] \qquad \text{for all } t \in [0, T] \tag{IV-27}$$

and, therefore,

$$\int_0^T h[u(t), t]\, dt \leqslant \int_0^T h[\hat{u}(t), t]\, dt \quad . \tag{IV-28}$$

Since $u \in U$, then $u(t) = \underline{\psi}^t(t)\, \underline{c}$. Using this fact and Eq. (IV-23) in Eq. (IV-28), we find that

$$\left[-\int_0^T \lambda_1(t)\, \underline{\psi}^t(t)\, dt\right] \underline{c} + \lambda_5 \underline{c}^t \left[\int_0^T \underline{\psi}(t)\, \underline{\psi}^t(t)\, dt\right] \underline{c}$$

$$\leqslant \left[-\int_0^T \lambda_1(t)\, \underline{\psi}^t(t)\, dt\right] \hat{\underline{c}} + \lambda_5 \hat{\underline{c}}^t \left[\int_0^T \underline{\psi}(t)\, \underline{\psi}^t(t)\, dt\right] \hat{\underline{c}} \tag{IV-29}$$

where $\underline{\hat{c}}$ leads to the optimal control $\hat{u}$. Since the basis functions were orthonormal, the matrix

$$\int_0^T \underline{\psi}(t)\, \underline{\psi}^t(t)\, dt = I$$

which is the $m \times m$ identity matrix. Letting

$$\underline{\alpha} = -\int_0^T \lambda_1(t)\, \underline{\psi}(t)\, dt \tag{IV-30}$$

we conclude that the optimum set of coefficients $\underline{\hat{c}}$ must maximize the function

$$\ell(\underline{c}) = \underline{\alpha}^t \underline{c} + \lambda_5 \underline{c}^t \underline{c} \quad . \tag{IV-31}$$

We have already shown that $\lambda_5 < 0$; therefore, the $\underline{c}$ which maximizes Eq. (IV-31) is

$$\underline{\hat{c}} = -\underline{\alpha}/2\lambda_5 \quad . \tag{IV-32}$$

But, from Eqs. (IV-30) and (IV-24),

$$\underline{\hat{c}} = \frac{\lambda_3}{2\lambda_5} \int_0^T p(t)\, \underline{\psi}(t)\, dt + \frac{\lambda_4}{2\lambda_5} \int_0^T p'(t)\, \underline{\psi}(t)\, dt \quad . \tag{IV-33}$$

We let

$$\underline{a} = \int_0^T p'(t)\, \underline{\psi}(t)\, dt \tag{IV-34a}$$

$$\underline{b} = \int_0^T p(t)\, \underline{\psi}(t)\, dt \tag{IV-34b}$$

and, therefore,

$$\underline{\hat{c}} = \frac{\lambda_4}{2\lambda_5} \underline{a} + \frac{\lambda_3}{2\lambda_5} \underline{b} \quad . \tag{IV-35}$$

Substituting this back into the defining equation for $u \in U$, we see that the optimal control is

$$\hat{u}(t) = \frac{\lambda_4}{2\lambda_5} \underline{\psi}^t(t)\, \underline{a} + \frac{\lambda_3}{2\lambda_5} \underline{\psi}^t(t)\, \underline{b} \quad . \tag{IV-36}$$

The corresponding state variable x_1 is

$$\hat{x}_1(t) = -\frac{\lambda_4}{2\lambda_5} \underline{\varphi}(t)\, \underline{a} - \frac{\lambda_3}{2\lambda_5} \underline{\varphi}(t)\, \underline{b} \tag{IV-37}$$

where

$$\underline{\varphi}^t(t) = \int_0^t \underline{\psi}^t(\sigma)\, d\sigma \quad . \tag{IV-38}$$

It is convenient to define new functions

$$\chi_a(t) = \underline{\psi}^t(t)\, \underline{a} \tag{IV-39a}$$

$$\chi_b(t) = \underline{\psi}^t(t)\, \underline{b} \tag{IV-39b}$$

$$\Phi_a(t) = \underline{\varphi}^t(t)\,\underline{a} \tag{IV-40a}$$

$$\Phi_b(t) = \underline{\varphi}^t(t)\,\underline{b} \tag{IV-40b}$$

and variables

$$\alpha_1 = \lambda_4/2\lambda_5 \tag{IV-41a}$$

$$\alpha_2 = \lambda_3/2\lambda_5 \quad . \tag{IV-41b}$$

Notice that while α_1 and α_2 are as yet unknown, the functions χ_a, χ_b, Φ_a, and Φ_b are completely determined by the given pulse p(t) and filter basis functions $\underline{\psi}(t)$. By using this new notation, Eqs. (IV-36) and (IV-37) become

$$\hat{u}(t) = \alpha_1\chi_a(t) + \alpha_2\chi_b(t) \tag{IV-42a}$$

$$\hat{x}_1(t) = -\alpha_1\Phi_a(t) - \alpha_2\Phi_b(t) \quad . \tag{IV-42b}$$

As before, we have the zero bias constraint of Eq. (IV-16)

$$\hat{x}_3(T) = \int_o^T p'(t)\,\hat{x}_1(t)\,dt = 0 \tag{IV-43}$$

which requires that

$$\alpha_1 \int_o^T p'(t)\,\Phi_a(t)\,dt + \alpha_2 \int_o^T p'(t)\,\Phi_b(t)\,dt = 0 \quad . \tag{IV-44}$$

By letting

$$\rho(T) = -\frac{\int_o^T p'(t)\,\Phi_a(t)\,dt}{\int_o^T p'(t)\,\Phi_b(t)\,dt} \tag{IV-45}$$

Eq. (IV-44) then requires that

$$\alpha_2 = \rho(T)\,\alpha_1 \tag{IV-46}$$

and, from Eqs. (IV-42), we have

$$\hat{u}(t) = [\chi_a(t) + \rho(T)\,\chi_b(t)]\,\alpha_1 \tag{IV-47a}$$

$$\hat{x}_1(T) = -[\Phi_a(t) + \rho(T)\,\Phi_b(t)]\,\alpha_1 \quad . \tag{IV-47b}$$

The minimum cost is given by

$$-\hat{x}_4^2(T)/\hat{x}_5(T) \tag{IV-48}$$

where now

$$\begin{aligned} \hat{x}_4(T) &= \int_o^T p''(t)\,\hat{x}_1(t)\,dt \\ &= -\alpha_1 \int_o^T p''(t)\,[\Phi_a(t) + \rho(T)\,\Phi_b(t)]\,dt \end{aligned} \tag{IV-49a}$$

$$\hat{x}_5(T) = \int_0^T \hat{u}^2(t)\, dt$$

$$= \alpha_1^2 \int_0^T [\chi_a(t) + \rho(T)\, \chi_b(t)]^2\, dt \quad . \qquad \text{(IV-49b)}$$

The minimum cost is therefore

$$\frac{\left\{\int_0^T p''(t)\, [\Phi_a(t) + \rho(T)\, \Phi_b(t)]\, dt\right\}^2}{\int_0^T [\chi_a(t) + \rho(T)\, \chi_b(t)]^2\, dt} \qquad \text{(IV-50)}$$

which is again independent of the unknown multiplier α_1. Therefore, we can set $\alpha_1 = 1$, in which case the optimum solutions are

$$\hat{u}(t) = \chi_a(t) + \rho(T)\, \chi_b(t) \qquad \text{(IV-51a)}$$

$$\hat{x}_1(t) = -\Phi_a(t) - \rho(T)\, \Phi_b(t) \quad . \qquad \text{(IV-51b)}$$

More importantly, from Eq. (IV-46) and the fact that $\alpha_1 = 1$, we have $\alpha_2 = \rho(T)$. Then, from Eqs. (IV-41),

$$\lambda_4/2\lambda_5 = 1 \qquad \text{(IV-52a)}$$

$$\lambda_3/2\lambda_5 = \rho(T) \quad . \qquad \text{(IV-52b)}$$

Finally, in Eq. (IV-35), the optimum weights are

$$\hat{\underline{c}} = \underline{a} + \rho(T)\, \underline{b}$$

where $\underline{a}$, $\underline{b}$, and $\rho(T)$ are defined in Eqs. (IV-34a), (IV-34b), and (IV-45), respectively.

In effect, what we have done is to project the matched filter onto the subspace spanned by the set of functions $\{\psi_k\}_{k=1}^{m}$, where the projection has been done with respect to a norm involving the mean-square estimation accuracy of the filter processor.

From this example, we can conclude that the methodology to be used when even more stringent physical realizability constraints are to be imposed is relatively straightforward.

Example 3: Resolution constraints, unrestricted U

In this case, the system is the same as for Example 1 with the additional sidelobe constraints, and the steps in the synthesis are quite similar. For this reason, the details are presented in the Appendix. Here, it was not possible to obtain a completely analytical solution for the optimum control because of the large number of constraints which had to be satisfied. However, we were able to reduce the variational problem to a nonlinear programming problem involving a set of n + 3 unknown multipliers. It should be comparatively easier to generate the solution to the optimization problem in n + 3 space rather than to deal directly with the general variational problem with which we started.

V. CONCLUSIONS

The purpose of this report has been to outline the methodology to be used in formulating the mismatched filter design problem. An expression for the range accuracy using such a filter

has been derived to take into account the estimate bias. In addition, the resolution requirements of the radar have been included by imposing constraints on the sidelobes of the compressed pulse. State-space methods were then used to develop an optimal control problem whose solution led to the impulse response which, in turn, would yield the best mean-squared range accuracy subject to preassigned sidelobe constraints. Ignoring the sidelobes and applying the maximum principle led to the classical matched filter. Control techniques have been used before to obtain the matched filter solution, but the optimization has always involved a rather artificially defined signal-to-noise ratio. Our approach takes the range estimation performance into account directly and exhibits an "optimum" blend of the communication and control systems disciplines.

Another example was studied in which the sidelobes were again ignored, but this time the set of admissible filters was restricted to a linear combination of specified basis functions. The optimum weights could be found analytically and these led to an optimum filter within a subclass of filters not necessarily containing the matched filter. Such a realization can be obtained using RC structures which are relatively simple to build.

Applying the maximum principle to the design problem with sidelobe constraints led to a nonlinear programming problem involving the set of $n + 3$ unknown multipliers. Implementing algorithms for solving this subsidiary optimization problem should be easier than dealing with the original variational problem.

In the most general problem in which sidelobes are taken into account and in which the class of admissible filters is restricted by physical considerations, numerical methods will have to be used. However, computational techniques for obtaining the optimal solutions are well-established in the control field. The state-space formulation for the filter design therefore leads to a technique for generating the optimum mismatched filter.

APPENDIX

We shall now derive the necessary conditions for the optimum mismatched filter designed for sidelobe reduction as well as for good range accuracy.[7] In this case, the state equations are precisely those derived at the end of Sec. III and are summarized below for convenience.

$$
\begin{aligned}
&\dot{x}_1(t) = -u(t) && x_1(0) = 0 \\
&\dot{x}_2(t) = p(t)\, x_1(t) && x_2(0) = 0 \\
&\dot{x}_3(t) = p'(t)\, x_1(t) && x_3(0) = 0 \\
&\dot{x}_4(t) = p''(t)\, x_1(t) && x_4(0) = 0 \\
&\dot{x}_5(t) = u^2(t) && x_5(0) = 0 \\
&\dot{y}_j(t) = p(t+\tau_j)\, x_1(t) && y_j(0) = 0 \qquad j = 1, 2, \ldots, n
\end{aligned}
\tag{A-1}
$$

$$
\begin{aligned}
&x_3(T) = 0 \\
&y_j^2(T) - \epsilon_j x_2^2(T) \leqslant 0 \qquad j = 1, 2, \ldots, n \quad .
\end{aligned}
\tag{A-2}
$$

Subject to the preceding conditions, we wish to find the control function u(t) which minimizes the functional $-x_4^2(T)/x_5(T)$. Notice that scaling the control function by an arbitrary constant has no effect on the terminal constraints or the cost functional. Therefore, an equivalent problem is to minimize $-x_4^2(T)$ subject to the constraints of Eqs. (A-1) and (A-2) and the additional constraint

$$x_5(T) = 1 \quad . \tag{A-3}$$

As before, we define the functions

$$
\begin{aligned}
&\Theta_o[\underline{x}(T)] = -x_4^2(T) \\
&\Theta_1[\underline{x}(T)] = x_3(T) \\
&\Theta_2[\underline{x}(T)] = x_5(T) - 1 \\
&\psi_j[\underline{x}(T), \underline{y}(T)] = y_j^2(T) - \epsilon_j x_2^2(T) \qquad j = 1, 2, \ldots, n \quad .
\end{aligned}
\tag{A-4}
$$

In this case, the Hamiltonian is

$$
\begin{aligned}
H[\underline{x}(t), \underline{y}(t), u(t), \underline{\lambda}(t), \underline{\mu}(t)] = {}& -\lambda_1(t)\, u(t) + \lambda_2(t)\, p(t)\, x_1(t) + \lambda_3(t)\, p(t)\, x_1(t) \\
&+ \lambda_4(t)\, p''(t)\, x_1(t) + \lambda_5(t)\, u^2(t) \\
&+ \sum_{j=1}^{m} \mu_j(t)\, p(t+\tau_j)\, y_i(t) \quad .
\end{aligned}
\tag{A-5}
$$

The costate variables satisfy the equations

$$\dot{\lambda}_k(t) = \frac{\partial H}{\partial x_k(t)} \tag{A-6}$$

which, in this case, are given by

$$\dot{\lambda}_1(t) = -p(t)\,\lambda_2(t) - p'(t)\,\lambda_3(t) - p''(t)\,\lambda_4(t) - \sum_{j=1}^{n} p(t + \tau_j)\,\mu_j(t) \tag{A-7a}$$

$$\dot{\lambda}_k(t) = 0 \qquad k = 2, \ldots, 5 \tag{A-7b}$$

$$\dot{\mu}_j(t) = 0 \qquad j = 1, 2, \ldots, n \quad . \tag{A-7c}$$

The terminal values of the costate variables are obtained from

$$\lambda_j(T) = \sum_{i=0}^{2} \alpha_i \frac{\partial \Theta_i[\underline{x}(T)]}{\partial x_j(T)} + \sum_{i=1}^{n} \nu_i \frac{\partial \psi_i[\underline{x}(T), \underline{y}(T)]}{\partial x_j(T)} \tag{A-8a}$$

$$\mu_j(T) = \sum_{i=1}^{n} \nu_i \frac{\partial \psi_i[\underline{x}(T), \underline{y}(T)]}{\partial y_j(T)} \tag{A-8b}$$

and, therefore,

$$\left.\begin{aligned}
&\lambda_1(T) = 0 && \lambda_4(T) = -2\alpha_o x_4(T) \\
&\lambda_2(T) = -2x_2(T) \sum_{i=1}^{n} \nu_i \epsilon_i && \mu_j(T) = 2\nu_j y_j(T) \qquad j = 1, 2, \ldots, n \quad . \\
&\lambda_3(T) = \alpha_1 &&
\end{aligned}\right\} \tag{A-9}$$

In addition, the constant multipliers ν_i are zero if

$$y_i^2(T) - \epsilon_i x_2^2(T) < 0$$

and $\nu_i \leqslant 0$ otherwise.[7] Therefore, we assume that the summations on j include only those terms corresponding to sidelobe constraints which are actually on the boundary. Now we combine Eqs. (A-7) and (A-9) to get

$$\left.\begin{aligned}
&\lambda_2(t) \equiv -2x_2(T) \sum_{i=1}^{n} \epsilon_i \nu_i \triangleq \lambda_2 && \lambda_5(t) \equiv \alpha_2 \triangleq \lambda_5 \\
&\lambda_3(t) \equiv \alpha_1 \triangleq \lambda_3 && \mu_j(t) \equiv 2y_j(T)\,\nu_j \triangleq \mu_j \qquad j = 1, \ldots, n \\
&\lambda_4(t) \equiv -2\alpha_o x_4(T) \triangleq \lambda_4 &&
\end{aligned}\right\} \tag{A-10}$$

and the equation for $\lambda_1(t)$ is simply

$$\dot{\lambda}_1(t) = -\lambda_2 p(t) - \lambda_3 p'(t) - \lambda_4 p''(t) - \sum_{j=1}^{n} \mu_j p(t + \tau_j) \tag{A-11}$$

with $\lambda_1(T) = 0$. This equation is easily integrated to give

$$\lambda_1(t) = \lambda_2 \int_t^T p(\sigma)\, d\sigma + \lambda_3[p(T) - p(t)] + \lambda_4[p'(T) - p'(t)]$$

$$+ \sum_{j=1}^{n} \mu_j \int_t^T p(\sigma + \tau_j)\, d\sigma \quad . \tag{A-12}$$

As before, we assume that

$$p(0) = p(T) = 0 \tag{A-13a}$$

$$p'(0) = p'(T) = 0 \tag{A-13b}$$

so that

$$\lambda_1(t) = -\lambda_4 p(t) - \lambda_3 p(t) + \lambda_2 \int_t^T p(\sigma)\, d\sigma + \sum_{j=1}^{n} \mu_j \int_t^T p(\sigma + \tau_j)\, d\sigma \quad . \tag{A-14}$$

The maximum principle states that the optimum control must maximize the Hamiltonian. Therefore, it is necessary that the function

$$h(u) = -\lambda_1(t)\, u(t) + \lambda_5 u^2(t) \tag{A-15}$$

be maximized for the optimum $u \in U$. We shall assume that U is arbitrary and that $\alpha_o < 0$, $\alpha_5 < 0$. (This must be the case from arguments given in Sec. IV.) Therefore, the optimum control is of the form $\hat{u}(t) = \lambda_1(t)/2\lambda_5$. Using Eq. (A-14), we conclude that

$$\hat{u}(t) = \frac{\lambda_4}{2\lambda_5} p'(t) - \frac{\lambda_4}{2\lambda_5} p(t) + \frac{\lambda_2}{2\lambda_5} \int_t^T p(\sigma)\, d\sigma + \sum_{j=1}^{n} \frac{\mu_j}{2\lambda_5} \int_t^T p(\sigma + \tau_j)\, d\sigma \quad . \tag{A-16}$$

It is convenient to use vector notation to express $\hat{u}(t)$, which we do by defining the n + 3 vectors $\underline{\alpha}$ and $\underline{z}(t)$:

$$\alpha_1 = -\lambda_4/2\lambda_5 \qquad z_1(t) = p'(t)$$

$$\alpha_2 = -\lambda_3/2\lambda_5 \qquad z_2(t) = p(t)$$

$$\alpha_3 = +\lambda_2/2\lambda_5 \qquad z_3(t) = \int_t^T p(\sigma)\, d\sigma$$

$$\alpha_{j+3} = \mu_j/2\lambda_5 \qquad z_{j+3}(t) = \int_t^T p(\sigma + \tau_j)\, d\sigma \qquad j = 1, 2, \ldots, n \quad . \tag{A-17}$$

The optimum control may then be written as

$$\hat{u}(t) = \underline{\alpha}^t \underline{z}(t) \tag{A-18}$$

where the superscript t denotes the vector transpose. We define another n + 3 vector $\underline{\omega}(t)$ by setting

$$\underline{\omega}(t) = -\int_o^t \underline{z}(\sigma)\, d\sigma \quad . \tag{A-19}$$

Then, from Eqs. (A-1), the optimum filter is

$$\hat{x}_1(t) = \underline{\alpha}^t \underline{\omega}(t) \quad . \tag{A-20}$$

It is now possible to solve for the terminal values of the remaining state-variables by substituting in Eqs. (A-1)

$$\hat{x}_2(T) = \int_o^T p(t)\, \hat{x}_1 dt = \underline{\alpha}^t \int_o^T p(t)\, \underline{\omega}(t)\, dt$$

$$\hat{x}_3(T) = \int_o^T p'(t)\, x_1(t)\, dt = \underline{\alpha}^t \int_o^T p'(t)\, \underline{\omega}(t)\, dt$$

$$\hat{x}_4(T) = \int_o^T p''(t)\, \hat{x}_1 dt = \underline{\alpha}^t \int_o^T p''(t)\, \underline{\omega}(t)\, dt$$

$$\hat{x}_5(T) = \int_o^T \hat{u}^2(t)\, dt = \underline{\alpha}^t \left[\int_o^T \underline{z}(t)\, \underline{z}'(t)\, dt\right] \underline{\alpha}$$

$$\hat{y}_j(T) = \int_o^T p(t + \tau_j)\, \hat{x}_1(t)\, dt = \underline{\alpha}^t \int_o^T p(t + \tau_j)\, \underline{\omega}(t)\, dt \quad . \tag{A-21}$$

It is convenient to introduce some additional notation:

$$\underline{\gamma}_{pw} = \int_o^T p(t)\, \underline{\omega}(t)\, dt$$

$$\underline{\gamma}_{p'w} = \int_o^T p'(t)\, \underline{\omega}(t)\, dt$$

$$\underline{\gamma}_{p''w} = \int_o^T p''(t)\, \underline{\omega}(t)\, dt$$

$$\underline{\gamma}_j = \int_o^T p(t + \tau_j)\, \underline{\omega}(t)\, dt \tag{A-22}$$

where all the above vectors have n + 3 components. In addition, we define the (n + 3) × (n + 3) matrix

$$\Gamma = \int_o^T \underline{z}(t)\, \underline{z}^t(t)\, dt \tag{A-23}$$

so that

$$\hat{x}_2(T) = \underline{\alpha}^t \underline{\gamma}_{pw}$$

$$\hat{x}_3(T) = \underline{\alpha}^t \underline{\gamma}_{p'w}$$

$$\hat{x}_4(T) = \underline{\alpha}^t \underline{\gamma}_{p''w}$$

$$\hat{x}_5(T) = \underline{\alpha}^t \Gamma \underline{\alpha}$$

$$\hat{y}_j(T) = \underline{\alpha}^t \underline{\gamma}_j \quad . \tag{A-24}$$

It is important to note that the vectors $\underline{\gamma}$ and matrix Γ are completely known quantities. Only the vector $\underline{\alpha}$ needs to be determined, and it must satisfy the constraints of Eqs. (A-2) and (A-3), namely,

$$\hat{x}_3(T) = 0$$

$$\hat{y}_j^2(T) - \epsilon_j \hat{x}_2^2(T) \leqslant 0$$

$$\hat{x}_5(T) = 1 \qquad\qquad j = 1, 2, \ldots, n \quad . \tag{A-25}$$

These become

$$\underline{\alpha}^t \underline{\gamma}_{p'w} = 0$$

$$\underline{\alpha}^t \underline{\gamma}_j \underline{\gamma}_j^t \underline{\alpha} - \epsilon_j \underline{\alpha}^t \underline{\gamma}_{pw} \underline{\alpha} \leqslant 0$$

$$\underline{\alpha}^t \Gamma \underline{\alpha} = 1 \quad . \tag{A-26}$$

These equations represent n + 2 constraints on the n + 3 variables $\alpha_1, \ldots, \alpha_{n+3}$. The remaining degree of freedom is used to minimize the cost functional $-\hat{x}_4^2(T)$, which in this case is

$$-\underline{\alpha}^t \underline{\gamma}_{p''w} \underline{\gamma}_{p''w}^t \underline{\alpha} \quad . \tag{A-27}$$

Now, we let

$$\Gamma_j = \underline{\gamma}_j \underline{\gamma}_j^t - \epsilon_j \underline{\gamma}_{pw} \underline{\gamma}_{pw}^t$$

$$\Gamma_o = \underline{\gamma}_{p''w} (\underline{\gamma}_{p''w})^t$$

and the original variational problem for the optimum mismatched filter reduces to the following nonlinear programming problem.[8]

Find the n + 3 vector $\underline{\alpha}$ which maximizes the function

$$f(\underline{\alpha}) = \underline{\alpha}^t \Gamma_o \underline{\alpha}$$

subject to the constraints

$$r_1(\alpha) = \underline{\alpha}^t \underline{\gamma}_{pw} = 0$$

$$r_2(\alpha) = \underline{\alpha}^t \Gamma \underline{\alpha} - 1 = 0$$

$$q_j(\alpha) = \underline{\alpha}^t \Gamma_j \underline{\alpha} \leqslant 0 \qquad j = 1, 2, \ldots, n \quad . \tag{A-28}$$

It may be that this nonlinear programming problem is easier to solve than the original variational problem.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

Optimal Mismatched Filter Design for Radar Ranging and Resolution

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Note

5. AUTHOR(S) *(Last name, first name, initial)*

McAulay, Robert J. and Johnson, John R.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
8 May 1969	30	8

8a. CONTRACT OR GRANT NO. AF 19(628)-5167 b. PROJECT NO. None c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1969-6 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-69-108

10. AVAILABILITY/LIMITATION NOTICES

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Department of the Army, Office Chief Research and Development

13. ABSTRACT

In a single-target radar environment, matched filters provide the maximum output signal-to-noise ratio for target detection and yield the minimum mean-squared error estimate of target range. In a multiple-target environment, the sidelobes of the compressed pulse must be considered in the system design because of the likelihood of false alarms. In this case, the signal processor uses weighting filters which are not matched to the transmitted waveform. In this report, expressions for the mean-squared range estimation error, the estimate bias, and the effects of the sidelobes are derived in terms of the impulse response of an arbitrary mismatched filter. We desire to find that impulse response which leads to an unbiased estimate having the minimum range estimate variance subject to preassigned resolution (i.e., sidelobe) constraints. This optimization problem is formulated in state-space in which the optimal control law is sought. Pontryagin's maximum principle is used to obtain necessary conditions for the optimum filter. When the sidelobe constraints are neglected, these conditions lead to the matched filter solution. In an attempt to synthesize the optimal filter for the general case, we set up a nonlinear programming problem involving the set of unknown Lagrange multipliers. This should be a computationally easier problem to solve than the original variational problem. An example is given which illustrates the methodology for synthesizing the optimum filter when the class of admissible controls (i.e., filters) is restricted by physical considerations. It is in this case that the real power of the state-space development is clearly demonstrated.

14. KEY WORDS

filter design
radar ranging
radar resolution
SNR
Gaussian noise